The University of New South Wales

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On the q-symmetric fractional calculus and application

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Contents

List of Figures

List of Tables

Thanks

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Contents

Introduction

Al-Salam started the fitting of the concept of q-fractional calculus. After that he ([12],[13]) and Agarwal [?] continued by studying certain q-fractional integrals and derivatives.Recently, perhaps due to the explosion in research with in the fractional calculus setting, new developments in this theory of fractional q-difference calculus were made specifically,

There are two types of quantum calculus: the q-calculus and the h-calculus. In this paper we are concerned with the q-calculus. the q-calculus based on the notion of the q-derivative

$$\frac{f(qx)-f(x)}{x(q-1)},$$

where q is a fixed number different from 1, $x \neq 0$ and f is a real function. In contrast to the classical derivative, which measures the rate of change of the function of an incremental translation of its argument, the q-derivative measures the rate of change with respect to a dilatation of its argument by a factor q. It is clear that if f is differentiable at $x \neq 0$, then

$$f'(x) = \lim_{q\to 1} \frac{f(qx)-f(x)}{x(q-1)},$$

for a fixed $q \in]0,1[$ and $x \neq 0$ the q-symmetric derivative of a function f at point t is defined by

$$\frac{f(qx)-f(q^{-1}x)}{x(q-q^{-1})},$$

the q-symmetric derivative has important properties for the q-exponential function which turns out to be not true with the usual derivative.

The basic theory of q-symmetric quantum calculus needs to be explored. The object of this paper is to define a fractional q-symmetric operator corresponding to the q-symmetric analog of $\int_0^x f(t)\widetilde{d}_q t$. Besides this we shall investigate the fundamental properties of this operator. A study of these fractional q-symmetric operators is expected to be of great importance in the development of the q-function theory, which plays an important role in combinatory analysis.

This thesis master is organized as follows.

In chapter 1 we state the basic notations, definitions and properties concerning fractional q-symmetric calculus (q-Gamma and q-Beta functions, Riemann-Liouville, Caputo, q-symmetric fractional integral and derivative), which are used throughout the thesis to obtain our results.

Next, in chapter 2, we will discuss the existence and uniqueness of the following nonlocal q-symmetric integral boundary value problem of nonlinear q-symmetric fractional differential equations

$$\begin{cases} (\widetilde{D}_{q,0}^{\alpha}u)(t) + f(q^{-\alpha}t, u(q^{-\alpha}t)) = 0, \quad t \in (0, q^{\alpha}), \\ u(0) = 0, \quad u(1) = \mu(\widetilde{I}_{q,0}^{\beta}u)(\eta), \end{cases}$$

where $q \in (0,1), 1 < \alpha \leq 2, 0 < \beta \leq 2, 0 < \eta < 1$, and $\mu > 0$ is a parameter, $\widetilde{D}_{q,0}^{\alpha}$ is the q-symmetric derivative of Riemann-Liouville type of order $\alpha, f : [0,1] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ is continuous.

Finally, in the last chapter we will study the following nonlocal q-symmetric integral boundary value problem of nonlinear q-symmetric fractional differential equations

$$\begin{cases} ({}^{c}\widetilde{D}_{q,0}^{\alpha}u)(t) + f(q^{-\alpha}t, u(q^{-\alpha}t)) = 0, \quad t \in (0, q^{\alpha}), \\ u(0) = 0, \quad u(1) = \mu(\widetilde{I}_{q,0}^{\beta}u)(\eta), \end{cases}$$

where $q \in (0,1), 1 < \alpha \leq 2, 0 < \beta \leq 2, 0 < \eta < 1$, and $\mu > 0$ is a parameter, ${}^{c}\widetilde{D}_{q,0}^{\alpha}$ is the q-symmetric derivative of Caputo type of order $\alpha, f : [0,1] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ is continuous.

Some existence and uniqueness results of solutions for the given problems are obtained by using the Banach contraction principle, Schauders, Schaufer fixed point theorem. Several examples are presented to illustrate the usefulness of our results.

Chapter 1

Basic definitions of q-Fractional Calculus

The first chapter will deal with the basic notations, definitions and properties concerning fractional q-symmetric calculus (q-Gamma and q-Beta functions, Riemann-Liouville, Caputo, q-symmetric fractional integral and derivative), which are used throughout the thesis.

1.1 Notations of q-symmetric fractional calculus

For a real parameter $q \in \mathbb{R}^+ \backslash \{1\}$, we introduce a q-real number $\overline{[a]}_q$by

$$\overline{[a]}_q = \frac{1 - q^{2a}}{1 - q^2}, \quad (a \in \mathbb{R}).$$

For a nonnegative integer n,

$$\overline{[0]}_q! = 1, \quad \overline{[n]}_q! = \overline{[n]}_q \overline{[n-1]}_q ... \overline{[1]}_q.$$

Also, the q-symmetric analog of the power $(a - b)^k$ is

$$\overline{(a-b)}^{(0)} = 1, \quad \overline{(a-b)}^{(k)} = \prod_{i=0}^{k-1} (a - bq^{2i+1}), \quad (k \in \mathbb{N}, a, b \in \mathbb{R}).$$

Their natural expansions to reals are

$$\overline{(a-b)}^{(\alpha)} = a^\alpha \frac{\prod_{i=0}^{\infty}(1 - \frac{b}{a} q^{2i+1})}{\prod_{i=0}^{\infty}(1 - \frac{b}{a} q^{2(i+\alpha)+1})}, \quad (\alpha \in \mathbb{R}, \ a \neq 0). \tag{1.1}$$

Definition 1 (q-Gamma function, see [7]) *The q-symmetric gamma function is defined by*

$$\begin{aligned}\widetilde{\Gamma}_q(x) &= \frac{\prod_{i=0}^{\infty}(1-q^{2i+2})}{\prod_{i=0}^{\infty}(1-q^{2(i+x-1)+2})}(1-q^2)^{1-x} \\ &= \overline{(1-q)}^{(x-1)}(1-q^2)^{1-x}, \quad (x \in \mathbb{R}\diagdown\{0,-1,-2,...\}). \end{aligned} \qquad (1.2)$$

Obviously,

$$\widetilde{\Gamma}_q(1) = \overline{(1-q)}^{(0)}(1-q^2)^0 = 1, \quad \widetilde{\Gamma}_q(x+1) = \overline{[x]}_q\widetilde{\Gamma}_q(x).$$

Definition 2 (q-Beta function, [7]) *For any $x, y > 0$,*

$$B_q(x,y) = \int_0^1 t^{(x-1)}(1-qt)^{(y-1)}d_qt.$$

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}.$$

Therefore,

$$B_{\overline{q}}(x,y) = \frac{\Gamma_{\overline{q}}(x)\Gamma_{\overline{q}}(y)}{\Gamma_{\overline{q}}(x+y)} = \frac{\widetilde{\Gamma}_q(x)\widetilde{\Gamma}_q(y)}{\widetilde{\Gamma}_q(x+y)}. \qquad (1.3)$$

1.2 The q-symmetric analogs of Cauchy's formulas

The basic q-symmetric integrals are defined through the relations

$$(\widetilde{I}_{q,0}f)(x) = \int_0^x f(t)\widetilde{d}_qt = x(1-q^2)\sum_{k=0}^{\infty} q^{2k}f(xq^{2k+1}), \qquad (1.4)$$

$$(\widetilde{I}_{q,a}f)(x) = \int_a^x f(t)\widetilde{d}_qt = \int_0^x f(t)\widetilde{d}_qt - \int_0^a f(t)\widetilde{d}_qt.$$

Definition 3 (see [3]) *Let f be a real function defined on I (I be a interval of $\mathbb{R}$).The q-symmetric difference operator of f is defined by*

$$(\widetilde{D}_qf)(x) = \frac{f(qx) - f(\frac{x}{q})}{x(q-\frac{1}{q})}, \quad (\widetilde{D}_qf)(0) = f'(0), \qquad (1.5)$$

We usually call $(\widetilde{D}_qf)$ the q-symmetric derivative of f.

Example 4 *Let f be a real function defined on I such that*

$$f(x) = x^2,$$

The q-symmetric differance operator of f is

$$\begin{aligned}(\widetilde{D}_q f)(x) &= \widetilde{D}_q x^2 \\ &= \frac{(qx)^2 - (\frac{x}{q})^2}{x(q - \frac{1}{q})} \\ &= \frac{x(q^2 - \frac{1}{q^2})}{(q - \frac{1}{q})}.\end{aligned}$$

We take $q = \frac{1}{2}$,

$$(\widetilde{D}_q f)(x) = \frac{15}{6} x.$$

The q-symmetric derivatives of higher order as

$$(\widetilde{D}_q^0 f)(x) = f(x), \;\; (\widetilde{D}_q^n f)(x) = (\widetilde{D}_q \widetilde{D}_q^{n-1} f)(x), \; n \in \mathbb{N}^+.$$

As for q-symmetric derivatives, we can define an operator $\tilde{I}_q^n$ by

$$(\widetilde{I}_{q,a}^0 f)(x) = f(x), \;\; (\widetilde{I}_{q,a}^n f)(x) = (\widetilde{I}_{q,a} \widetilde{I}_{q,a}^{n-1} f)(x), \; n \in \mathbb{N}^+. \tag{1.6}$$

Corollary 5 (see [3]) *If $f : I \to \mathbb{R}$ is continuous at 0, then for $s \in [a, b]$ the series*

$$\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1} s),$$

is uniformly convergent on I and f is q-symmetric integrable on $[a, b]$.

Theorem 6 (see [3]) *(Fundamental Theorem of the q-symmetric Integral calculus). Assume that $f : I \to \mathbb{R}$ is continuous at 0 and for each $x \in I$, define*

$$F(x) := (\widetilde{I}_{q,0} f)(t) = \int_0^x f(t) \widetilde{d}_q t.$$

Then $(\widetilde{I}_{q,0} f)$ is continuous at 0, with

$$(\widetilde{D}_q \widetilde{I}_{q,0} f)(x) = f(x), \;\; (\widetilde{I}_{q,0} \widetilde{D}_q f)(x) = f(x) - f(0). \tag{1.7}$$

Furthermore, $\widetilde{D}_q[F](x)$ exists for every $x \in I$ with $\widetilde{D}_q[F](x) = f(x)$. Conversely,

$$\int_a^b \widetilde{D}_q[f](t)\widetilde{d}_q t = f(b) - f(a)$$

for all $a, b \in I$.

Proof. *By Corollary 5, the function F is continuous at 0, if $x \in I\backslash\{0\}$ then*

$$\begin{aligned}
\widetilde{D}_q(\int_0^x f(t)\widetilde{d}_q t) &= \frac{\int_0^{qx} f(t)\widetilde{d}_q t - \int_0^{q^{-1}x} f(t)\widetilde{d}_q t}{(q-q^{-1})x} \\
&= \frac{q}{(q^2-1)x}[(1-q^2)qx\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1}qx) \\
&\quad -(1-q^2)q^{-1}x\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1}q^{-1}x)] \\
&= \sum_{n=0}^{+\infty} q^{2n} f(q^{2n}x) - \sum_{n=0}^{+\infty} q^{2n+2} f(q^{2n+2}x) \\
&= \sum_{n=0}^{+\infty} [q^{2n} f(q^{2n}x) - q^{2(n+1)} f(q^{2(n+1)}x) \\
&= f(x).
\end{aligned}$$

If $x = 0$, then

$$\begin{aligned}
\widetilde{D}_q[F](0) &= \lim_{h\to 0}\frac{F(h)-F(0)}{h} \\
&= \lim_{h\to 0}\frac{1}{h}(1-q^2)h\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1}h) \\
&= \lim_{h\to 0}(1-q^2)\sum_{n=0}^{+\infty} q^{2n} f(q^{2n+1}h) \\
&= (1-q^2)\sum_{n=0}^{+\infty} q^{2n} f(0) \quad \textit{(by the continuity of } f \textit{ at } 0\textit{)} \\
&= (1-q^2)\frac{1}{1-q^2} f(0) \\
&= f(0).
\end{aligned}$$

Finally, since for $x \in I$,

$$\begin{aligned}
\int_0^x \widetilde{D}_q[f](t)\widetilde{d}_q t &= (1-q^2)x\sum_{n=0}^{+\infty} q^{2n}\widetilde{D}_q[f](q^{2n+1}x) \\
&= (1-q^2)x\sum_{n=0}^{+\infty} q^{2n}\frac{f(qq^{2n+1}x)-f(q^{-1}q^{2n+1}x)}{(q^{-1}-q)(q^{2n+1}x)} \\
&= \sum_{n=0}^{+\infty}[f(q^{2n}x)-f(q^{2(n+1)}x)] \\
&= f(x)-f(0),
\end{aligned}$$

we have

$$\begin{aligned}
\int_a^b \widetilde{D}_q[f](t)\widetilde{d}_q t &= \int_0^b \widetilde{D}_q[f](t)\widetilde{d}_q t - \int_0^a \widetilde{D}_q[f](t)\widetilde{d}_q t \\
&= f(b)-f(a).
\end{aligned}$$

■

Theorem 7 (see [3]) *Let $f, g : I \to \mathbb{R}$ be q-symmetric integrable on $I, a, b, c \in I$ and $\alpha, \beta \in \mathbb{R}$. Then*
1. $\int_a^a f(t)\widetilde{d}_q t = 0$
2. $\int_a^b f(t)\widetilde{d}_q t = -\int_b^a f(t)\widetilde{d}_q t$;
3. $\int_a^b f(t)\widetilde{d}_q t = \int_a^c f(t)\widetilde{d}_q t + \int_c^b f(t)\widetilde{d}_q t$;
4. $\int_a^b (\alpha f + \beta g)(t)\widetilde{d}_q t = \alpha \int_a^b f(t)\widetilde{d}_q t + \beta \int_a^b g(t)\widetilde{d}_q t$;
5. If $\widetilde{D}_q[f]$ and $\widetilde{D}_q[g]$ are continuous at 0, then

$$\int_a^b f(q^{-1}t)\widetilde{D}_q[g](t)\widetilde{d}_q t = f(t)g(t)\,\big|_a^b - \int_a^b \widetilde{D}_q[f](t)g(qt)\widetilde{d}_q t.$$

We call this formula q-symmetric integration by parts.

Proof. Properties 1–4 are trivial. Property 5 follows from Theorems 6:

$$\widetilde{D}_q[fg](t) = \widetilde{D}_q[f](t)g(qt) + f(q^{-1}t)\widetilde{D}_q[g](t)$$

$$\Longrightarrow f(q^{-1}t)\widetilde{D}_q[g](t) = \widetilde{D}_q[fg](t) - \widetilde{D}_q[f](t)g(qt)$$

$$\Longrightarrow \int_a^b f(q^{-1}t)\widetilde{D}_q[g](t)\widetilde{d}_q t = f(t)g(t)\,\big|_a^b - \int_a^b \widetilde{D}_q[f](t)g(qt)\widetilde{d}_q t.$$

■

Remark 8 (see [3]) *In general it is not true that if f is a positive function on $[a,b]$, then*

$$\int_a^b f(t)\widetilde{d}_q t \geq 0.$$

Example 9 *Consider the function f defined in $[-1,1]$ by*

$$f(t) = \begin{cases} 1 & if \quad t = \frac{1}{2} \\ 6 & if \quad t = \frac{1}{6} \\ 0 & if \quad t \in [-1,1] \setminus \left\{\frac{1}{6}, \frac{1}{2}\right\} \end{cases}$$

For $q = \frac{1}{2}$ this function is q-symmetric integrable because is continuous at $t = 0$ and

$$\begin{aligned} \int_{\frac{1}{3}}^1 f(t)\widetilde{d}_q t &= \int_0^1 f(t)\widetilde{d}_q t - \int_0^{\frac{1}{3}} f(t)\widetilde{d}_q t \\ &= \frac{3}{4}\sum_{n=0}^{+\infty}(\frac{1}{2})^{2n} f\left((\frac{1}{2})^{2n+1}\right) - \frac{3}{4}(\frac{1}{3})\sum_{n=0}^{+\infty}(\frac{1}{2})^{2n} f\left(\frac{1}{3}.(\frac{1}{2})^{2n+1}\right) \\ &= \frac{3}{4} \times 1 - \frac{1}{4} \times 6 \\ &= -\frac{3}{4}. \end{aligned}$$

Lemma 10 *Using (1.1) and (1.4), we may obtain the very useful examples of the q-symmetric derivatives of the next functions:*

$$_x\widetilde{D}_q \overline{(x-a)}^{(\alpha)} = \overline{[\alpha]}_q \overline{(\frac{x}{q} - a)}^{(\alpha-1)}, \tag{1.8}$$

$$_x\widetilde{D}_q \overline{(a - \frac{x}{q})}^{(\alpha)} = -\overline{[\alpha]}_q \overline{(a-x)}^{(\alpha-1)}, \tag{1.9}$$

Next, we consider the form of the multiple q-symmetric integration as follows:

$$(\widetilde{I}_q^n f)(x) = \int_0^x \widetilde{d}_q t \int_0^t \widetilde{d}_q t_{n-1} \int_0^{t_{n-1}} \widetilde{d}_q t_{n-2} ... \int_0^{t_2} \widetilde{d}_q t_1. \tag{1.10}$$

Theorem 11 (see [7]) *The form of the multiple q-symmetric integration (1.10) is equality to*

$$(\widetilde{I}_{q,0}^n f)(x) = \frac{1}{\overline{[n-1]}_q!} q^{\binom{n}{2}} \int_0^x \overline{(x-\tau)}^{(n-1)} f(q^{n-1}\tau)\widetilde{d}_q \tau, \tag{1.11}$$

where

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

1.3 The fractional q-symmetric integral operator

Definition 12 *the fractional q-symmetric integral operator,*

$$(\widetilde{I}^{\alpha}_{q,0}f)(x) = \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^x \overline{(x-\tau)}^{(\alpha-1)} f(q^{\alpha-1}\tau)\widetilde{d}_q\tau, \ (\alpha \in \mathbb{R}^+), \tag{1.12}$$

where

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad (k \in \mathbb{N}).$$

To prove the semigroup property of the fractional q-symmetric integral, we need Lemma 13.

Lemma 13 (see [7]) *For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid*

$$\sum_{n=0}^{\infty} \frac{(1-\mu q^{1-n})^{(\alpha-1)}(1-\mu q^{1+n})^{(\beta-1)}}{(1-q)^{(\alpha-1)}(1-q)^{(\beta-1)}} q^{\alpha n} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}},$$

where

$$(a-b)^{(\alpha)} = a^{\alpha} \frac{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^i)}{\prod_{i=0}^{\infty}(1-\frac{b}{a}q^{\alpha+1})}, \quad (a, b \in \mathbb{R}, a \neq 0).$$

Theorem 14 ([7]) *Let $\alpha, \beta \in \mathbb{R}^+$.The fractional q-symmetric integration has the following semigroup property:*

$$(\widetilde{I}^{\alpha}_{q,0}\widetilde{I}^{\beta}_{q,0}f)(x) = (\widetilde{I}^{\alpha+\beta}_{q,0}f)(x).$$

Theorem 15 ([7]) *For $\alpha \in \mathbb{R}^+$, the following identity is valid*

$$(\widetilde{I}^{\alpha}_{q,0}f)(x) = (\widetilde{I}^{\alpha+1}_{q,0}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} x^{\alpha}.$$

Proof. Using the q-symmetric integration by parts and (1.9), we obtain

$$\begin{aligned}
(\widetilde{I}_{q,0}^{\alpha}f)(x) &= \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^x \overline{(x-\tau)}^{(\alpha-1)}f(q^{\alpha-1}\tau)\widetilde{d}_q\tau \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha)\overline{[\alpha]}_q}q^{\binom{\alpha}{2}}(-\int_0^x f(q^{\alpha-1}\tau)\widetilde{d}_q\overline{(x-q^{-1}\tau)}^{(\alpha)}) \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}(-\overline{(x-q^{-1}\tau)}^{(\alpha)}f(q^{\alpha}\tau)\mid_0^x \\
&\quad +\int_0^x q^{\alpha}\overline{(x-\tau)}^{(\alpha)}\widetilde{D}_qf(q^{\alpha}\tau)\widetilde{d}_q\tau) \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}(-\overline{(x-q^{-1}\tau)}^{(\alpha)}f(q^{\alpha}x) \\
&\quad +x^{\alpha}f(0)+\int_0^x q^{\alpha}\overline{(x-\tau)}^{(\alpha)}\widetilde{D}_qf(q^{\alpha}\tau)\widetilde{d}_q\tau) \\
&= (\widetilde{I}_{q,0}^{\alpha+1}\widetilde{D}_qf)(x)+\frac{f(0)}{\widetilde{\Gamma}_q(\alpha+1)}q^{\binom{\alpha}{2}}x^{\alpha}.
\end{aligned}$$

■

1.4 The fractional q-symmetric derivative

1.4.1 The fractional q-symmetric derivative of Riemann-Liouville

Definition 16 *the fractional q-symmetric dervative of Riemann-Liouville type of a function $f(x)$ define by*

$$(\widetilde{D}_{q,0}^{\alpha}f)(x)=\begin{cases}(\widetilde{I}_{q,0}^{-\alpha}f)(x), & \alpha<0,\\ f(x), & \alpha=0,\\ (\widetilde{D}_{q,0}^{[\alpha]}\widetilde{I}_{q,0}^{[\alpha]-\alpha}f)(x), & \alpha>0.\end{cases} \tag{1.13}$$

Here $[\alpha]$ denotes the smallest integer greater than or equal to α.

Lemma 17 ([7]) *For $\alpha\in\mathbb{R}^+$ and $\lambda\in(-1,\infty)$, the following is valid*

$$\begin{aligned}
(i)\ \widetilde{I}_{q,0}^{\alpha}x^{\lambda} &= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+\alpha+1)}q^{\binom{\alpha}{2}+\lambda\alpha}x^{\lambda+\alpha}. \\
(ii)\ \widetilde{D}_{q,0}^{\alpha}x^{\lambda} &= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}q^{\binom{-\alpha}{2}-\lambda\alpha}x^{\lambda-\alpha},\ (\lambda-\alpha+1\neq 0).
\end{aligned}$$

Proof. (i) For $\lambda \neq 0$, according to (1.12), we have

$$
\begin{aligned}
\widetilde{I}_{q,0}^{\alpha} x^{\lambda} &= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^x \overline{(x-\tau)}^{(\alpha-1)} (q^{\alpha-1}\tau) \widetilde{d}_q \tau \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} x^{\alpha+\lambda} q^{\lambda(\alpha-1)} \int_0^x \overline{(1-\frac{\tau}{x})}^{(\alpha-1)} \left(\frac{\tau}{x}\right)^{\lambda} \widetilde{d}_q \left(\frac{\tau}{x}\right) \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} x^{\alpha+\lambda} q^{\lambda(\alpha-1)} \int_0^1 \overline{(1-s)}^{(\alpha-1)} s^{\lambda} \widetilde{d}_q s.
\end{aligned}
$$

Let $q^2 = \overline{q}$, by (1.4), (1.2), and Definition 2, we get

$$
\begin{aligned}
\int_0^1 \overline{(1-s)}^{(\alpha-1)} s^{\lambda} \widetilde{d}_q s &= x(1-q^2) \sum_{k=0}^{\infty} q^{2k} \overline{(1-q^{2k+1})}^{(\alpha-1)} q^{\lambda(2k+1)} \\
&= q^{\lambda}(1-q^2) \sum_{k=0}^{\infty} q^{2k} \prod_{i=0}^{\infty} \frac{1-q^{2(k+i+1)}}{1-q^{2(k+i+\alpha)}} q^{2\lambda k} \\
&= \overline{q}^{\frac{\lambda}{2}} (1-\overline{q}) \sum_{k=0}^{\infty} \overline{q}^k \prod_{i=0}^{\infty} \frac{1-\overline{q}^{k+i+1}}{1-\overline{q}^{k+i+\alpha}} \overline{q}^{\lambda k} \\
&= \overline{q}^{\frac{\lambda}{2}} (1-\overline{q}) \sum_{k=0}^{\infty} \overline{q}^k (1-\overline{q}^{k+1}) \overline{q}^{\lambda k} \\
&= \overline{q}^{\frac{\lambda}{2}} \int_0^1 (1-\overline{q}x)^{(\alpha-1)} x^{\lambda} d_{\overline{q}} x \\
&= \overline{q}^{\frac{\lambda}{2}} B_{\overline{q}}(\lambda+1, \alpha) \\
&= q^{\lambda} \frac{\widetilde{\Gamma}_q(\alpha)\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+\alpha+1)}.
\end{aligned}
$$

Hence, we obtain the required formula for $\widetilde{I}_{q,0}^{\alpha} x^{(\lambda)}$when $\lambda \neq 0$.
If $\lambda = 0$, then using (1.9), we have

$$
\begin{aligned}
\widetilde{I}_{q,0}^{\alpha} 1 &= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^x \overline{(x-\tau)}^{(\alpha-1)} \widetilde{d}_q \tau \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^x \frac{\widetilde{D}_q \overline{(x-\frac{\tau}{q})}^{(\alpha)}}{-\overline{[\alpha]}_q} \widetilde{d}_q \tau \\
&= \frac{-1}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} \int_0^x \widetilde{D}_q \overline{(x-\frac{\tau}{q})}^{(\alpha)} \widetilde{d}_q \tau \\
&= \frac{1}{\widetilde{\Gamma}_q(\alpha+1)} q^{\binom{\alpha}{2}} x^{\alpha}.
\end{aligned}
$$

(ii) By (i) and (1.13) and (1.8), we get

$$\begin{aligned}
\widetilde{D}_{q,0}^{\alpha}(x^{\lambda}) &= \widetilde{D}_q^{[\alpha]}\widetilde{I}_{q,0}^{[\alpha]-\alpha}(x^{\lambda}) \\
&= \widetilde{D}_q^{[\alpha]}\left(\frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}x^{\lambda+[\alpha]-\alpha}\right) \\
&= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}\widetilde{D}_q^{[\alpha]}(x^{\lambda+[\alpha]-\alpha}) \\
&= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}q^{\binom{[\alpha]-\alpha}{2}}q^{\lambda([\alpha]-\alpha)}q^{-\binom{[\alpha]+1}{2}-[\alpha](\lambda-\alpha-1)} \\
&\quad\times\frac{\widetilde{\Gamma}_q(\lambda+[\alpha]-\alpha+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}x^{\lambda-\alpha} \\
&= \frac{\widetilde{\Gamma}_q(\lambda+1)}{\widetilde{\Gamma}_q(\lambda-\alpha+1)}q^{\binom{-\alpha}{2}-\lambda\alpha}x^{\lambda-\alpha}.
\end{aligned}$$

■

Theorem 18 ([7]) *For* $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, *the following is valid*

$$(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f)(x) = (\widetilde{D}_{q,0}^{\alpha+1}f)(x).$$

Proof. We consider three cases. For $\alpha \le -1$, according to Theorem 15 and (1.7), we have

$$\begin{aligned}
(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f)(x) &= (\widetilde{D}_q\widetilde{I}_{q,0}^{-\alpha}f)(x) \\
&= (\widetilde{D}_q\widetilde{I}_{q,0}^{1-\alpha-1}f)(x) \\
&= (\widetilde{I}_{q,0}^{-(\alpha+1)}f)(x) \\
&= (\widetilde{D}_{q,0}^{(\alpha+1)}f)(x).
\end{aligned}$$

In th case $-1 < \alpha < 0$, we obtain

$$\begin{aligned}
(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f)(x) &= (\widetilde{D}_q\widetilde{I}_{q,0}^{-\alpha}f)(x) \\
&= (\widetilde{D}_q\widetilde{I}_{q,0}^{1-(\alpha+1)}f)(x) \\
&= (\widetilde{D}_q^{\alpha+1}f)(x).
\end{aligned}$$

For $\alpha > 0$, we get

$$\begin{aligned}
(\widetilde{D}_q\widetilde{D}_{q,0}^{\alpha}f)(x) &= (\widetilde{D}_q\widetilde{D}_q^{[\alpha]}\widetilde{I}_{q,0}^{[\alpha]-\alpha}f)(x) \\
&= (\widetilde{D}_{q,0}^{\alpha+1}f)(x).
\end{aligned}$$

■

Theorem 19 ([7]) *For $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, the following is valid*

$$(\widetilde{D}_q \widetilde{D}_{q,0}^{\alpha} f)(x) = (\widetilde{D}_{q,0}^{\alpha} \widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)} q^{\binom{-(\alpha+1)}{2}} x^{-\alpha-1}.$$

Proof. Let us consider two cases. For $\alpha < 0$, we have

$$\begin{aligned}
(\widetilde{D}_q \widetilde{D}_{q,0}^{\alpha} f)(x) &= (\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} f)(x) \\
&= \widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} [(\widetilde{I}_{q,0} \widetilde{D}_q f)(x) + f(0)] \\
&= (\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} \widetilde{I}_{q,0} \widetilde{D}_q f)(x) + f(0)(\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} 1) \\
&= (\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha+1} \widetilde{D}_q f)(x) + f(0) q^{\binom{-\alpha}{2}} q^{-(\alpha+1)} \frac{1}{\widetilde{\Gamma}_q(-\alpha)\overline{[-\alpha]}_q} \widetilde{D}_q x^{-\alpha} \\
&= (\widetilde{D}_{q,0}^{\alpha} \widetilde{D}_q f)(x) + f(0) q^{\binom{-\alpha}{2}} q^{-(\alpha+1)} \frac{1}{\widetilde{\Gamma}_q(-\alpha)} x^{-\alpha-1} \\
&= (\widetilde{D}_{q,0}^{\alpha} \widetilde{D}_q f)(x) + f(0) q^{\binom{-(\alpha+1)}{2}} \frac{1}{\widetilde{\Gamma}_q(-\alpha)} x^{-\alpha-1}.
\end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$, such that $\alpha \in (l, l+1)$. Then applying a similar procedure, we get

$$\begin{aligned}
(\widetilde{D}_q \widetilde{D}_{q,0}^{\alpha} f)(x) &= (\widetilde{D}_q \widetilde{D}_q^{l+1} \widetilde{I}_{q,0}^{l+1-\alpha} f)(x) \\
&= \widetilde{D}_q^{l+2} \widetilde{I}_{q,0}^{l+1-\alpha} [(\widetilde{I}_{q,0} \widetilde{D}_q f)(x) + f(0)] \\
&= (\widetilde{D}_q^{l+2} \widetilde{I}_{q,0}^{l+2-\alpha} \widetilde{D}_q f)(x) + f(0) \frac{q^{\binom{l+1-\alpha}{2}}}{\widetilde{\Gamma}_q(l+1-\alpha)} \left(\widetilde{D}_q^{l+2} \int_0^x \overline{(x-\tau)}^{(l-\alpha)} \widetilde{d}_q \tau \right) \\
&= (\widetilde{D}_q^{l+2} \widetilde{I}_{q,0}^{l+2-\alpha} \widetilde{D}_q f)(x) + f(0) \frac{q^{\binom{l+1-\alpha}{2}}}{\widetilde{\Gamma}_q(l+2-\alpha)} (\widetilde{D}_q^{l+2} x^{l+1-\alpha}) \\
&= (\widetilde{D}_{q,0}^{\alpha} \widetilde{D}_q f)(x) + f(0) q^{\binom{-(\alpha+1)}{2}} \frac{1}{\widetilde{\Gamma}_q(-\alpha)} x^{-\alpha-1}.
\end{aligned}$$

■

Theorem 20 ([7]) *For $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, the following is valid*

$$(\widetilde{D}_{q,0}^{\alpha} \widetilde{I}_{q,0}^{\alpha} f)(x) = f(x).$$

Proof.

$$\begin{aligned}
(\widetilde{D}_{q,0}^{\alpha} \widetilde{I}_{q,0}^{\alpha} f)(x) &= (\widetilde{D}_q^{[\alpha]} \widetilde{I}_{q,0}^{[\alpha]-\alpha} \widetilde{I}_{q,0}^{\alpha} f)(x) \\
&= (\widetilde{D}_q^{[\alpha]} \widetilde{I}_{q,0}^{[\alpha]} f)(x) \\
&= f(x).
\end{aligned}$$

■

Theorem 21 ([7]) *Let* $\alpha \in (0,1)$. *Then*

$$(\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f)(x) = f(x) + Kx^{\alpha-1}.$$

Proof. Let

$$A(x) = (\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f)(x) - f(x).$$

Apply $\widetilde{D}_{q,0}^{\alpha}$ to both sides of the above expression, and using Theorem 20, we get

$$\begin{aligned}(\widetilde{D}_{q,0}^{\alpha})A(x) &= (\widetilde{D}_{q,0}^{\alpha}\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f)(x) - (\widetilde{D}_{q,0}^{\alpha}f)(x)\\ &= (\widetilde{D}_{q,0}^{\alpha}f)(x) - (\widetilde{D}_{q,0}^{\alpha}f)(x)\\ &= 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_0^x \overline{(x-\tau)}^{(-\alpha)}\left(q^{-\alpha}\tau\right)^{\alpha-1}\widetilde{d}_q\tau\\ = \;& x(1-q^2)\sum_{m=0}^{\infty} q^{2m}\overline{(x-q^{2m+1}x)}^{(-\alpha)}(q^{-\alpha}q^{2m+1}x)^{\alpha-1}\\ = \;& (1-q^2)\sum_{m=0}^{\infty} q^{2m}\overline{(1-q^{2m+1})}^{(-\alpha)}(q^{-\alpha}q^{2m+1})^{\alpha-1}.\end{aligned}$$

Using the above form and according to (1.1), (1.12), we obtain

$$\begin{aligned}\widetilde{D}_{q,0}^{\alpha}x^{\alpha-1} &= \widetilde{D}_q\widetilde{I}_{q,0}^{1-\alpha}x^{\alpha-1}\\ &= \widetilde{D}_q q^{\binom{1-\alpha}{2}}\frac{1}{\widetilde{\Gamma}_q(1-\alpha)}\int_0^x \overline{(x-\tau)}^{(-\alpha)}(q^{-\alpha}\tau)^{\alpha-1}\widetilde{d}_q\tau\\ &= 0.\end{aligned}$$

Hence $A(x) = Kx^{\alpha-1}$. ■

Theorem 22 ([7]) *Let* $\alpha \in (N-1, N]$. *Then for some constants* $c_i \in \mathbb{R}$, $i = 1,2,...,N$, *the following equality holds*

$$(\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f)(x) = f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + ... + c_Nx^{\alpha-N}. \tag{1.14}$$

Proof. By Theorem 15 and Theorem 21, we have

$$
\begin{aligned}
(\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q,0}^{\alpha}f)(x) &= (\widetilde{I}_{q,0}^{\alpha}\widetilde{D}_{q}^{N}\widetilde{I}_{q,0}^{N-\alpha}f)(x) \\
&= (\widetilde{I}_{q,0}^{\alpha-1}\widetilde{D}_{q}^{N-1}\widetilde{I}_{q,0}^{N-\alpha}f)(x) - q^{\binom{\alpha-2}{2}}\frac{\widetilde{D}_{q}^{N-1}\widetilde{I}_{q,0}^{N-\alpha}f(0)}{\widetilde{\Gamma}_{q}(\alpha-1)}x^{\alpha-2} \\
&\quad -q^{\binom{1-\alpha}{2}}\frac{\widetilde{D}_{q}^{N}\widetilde{I}_{q,0}^{N-\alpha}f(0)}{\widetilde{\Gamma}_{q}(\alpha)}x^{\alpha-1} \\
&= \ldots \\
&= (\widetilde{I}_{q,0}^{\alpha-N}\widetilde{D}_{q,0}^{\alpha-N+1}f)(x) - q^{\binom{\alpha-N+1}{2}}\frac{\widetilde{D}_{q}\widetilde{I}_{q,0}^{N-\alpha}f(0)}{\widetilde{\Gamma}_{q}(\alpha-N+2)}x^{\alpha-N+1} - \ldots \\
&\quad -q^{\binom{\alpha-1}{2}}\frac{\widetilde{D}_{q}^{N-1}\widetilde{I}_{q,0}^{N-\alpha}f(0)}{\widetilde{\Gamma}_{q}(\alpha)}x^{\alpha-1} \\
&= f(x) + c_1x^{\alpha-1} + c_2x^{\alpha-2} + \ldots + c_Nx^{\alpha-N}.
\end{aligned}
$$

■

1.4.2 The fractional q-symmetric derivative of caputo type

If we change the order of operators, we can introduce another type of fractional q-derivative.

Definition 23 *The fractional q-symmetric derivative of caputo type is*

$$
({}^{c}\widetilde{D}_{q,0}^{\alpha}f)(x) = \begin{cases} \left(\widetilde{I}_{q,0}^{-\alpha}f\right)(x), & \alpha < 0, \\ f(x), & \alpha = 0, \\ (\widetilde{I}_{q,0}^{[\alpha]-\alpha}\widetilde{D}_{q}^{[\alpha]}f)(x), & \alpha > 0. \end{cases} \tag{1.15}
$$

Here $[\alpha]$ denotes the smallest integer greater than or equal to α.

Theorem 24 ([7]) *For $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, and $x > 0$, the following is valid*

$$
({}^{c}\widetilde{D}_{q,0}^{\alpha+1}f)(x) - ({}^{c}\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_{q}f)(x) = \begin{cases} \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)}q^{\left(-\binom{\alpha+1}{2}\right)}x^{-\alpha-1}, & \alpha \le -1, \\ 0, & \alpha > -1. \end{cases} \tag{1.16}
$$

Proof. Clearly, (1.16) holds for $\alpha = -1$. Next, we will consider three cases.
(i) If $\alpha < -1$,according to (1.15),(1.9) and (1.7), we have

$$\begin{aligned}
({}^c\widetilde{D}_{q,0}^{\alpha+1} f)(x) &= \left(\widetilde{I}_{q,0}^{-\alpha-1} f\right)(x) \\
&= \widetilde{I}_{q,0}^{-\alpha-1}(\widetilde{I}_{q,0}\widetilde{D}_q f(x) + f(0)) \\
&= (\widetilde{I}_{q,0}^{-\alpha}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)} q^{\binom{-(\alpha+1)}{2}} x^{-\alpha-1} \\
&= ({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)} q^{\binom{-(\alpha+1)}{2}} x^{-\alpha-1}.
\end{aligned}$$

(ii) If $-1 < \alpha \leq 0$, we obtain

$$\begin{aligned}
({}^c\widetilde{D}_{q,0}^{\alpha+1} f)(x) &= (\widetilde{I}_{q,0}^{1-(\alpha+1)}\widetilde{D}_q f)(x) \\
&= (\widetilde{I}_{q,0}^{-\alpha}\widetilde{D}_q f)(x) \\
&= ({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f)(x).
\end{aligned}$$

(iii) If $\alpha > 0$, we assume $\alpha = n+\varepsilon$, $n \in \mathbb{N}_0, 0 < \varepsilon < 1$, then $\alpha+1 \in (n+1, n+2)$, so we obtain

$$\begin{aligned}
({}^c\widetilde{D}_{q,0}^{\alpha+1} f)(x) &= (\widetilde{I}_{q,0}^{1-\varepsilon}\widetilde{D}_q^{n+2} f)(x) \\
&= (\widetilde{I}_{q,0}^{1-\varepsilon}\widetilde{D}_q^{n+1}\widetilde{D}_q f)(x) \\
&= ({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f)(x).
\end{aligned}$$

■

Theorem 25 ([7]) *For $\alpha \in \mathbb{R}\backslash\mathbb{N}_0$, and $x > 0$, the following is valid*

$$(\widetilde{D}_q {}^c\widetilde{D}_{q,0}^{\alpha} f)(x) - ({}^c\widetilde{D}_{q,0}^{\alpha+1} f)(x) = \begin{cases} \frac{\widetilde{D}_q^{[\alpha]} f(0)}{\widetilde{\Gamma}_q([\alpha]-\alpha)} q^{\binom{[\alpha]-(\alpha+1)}{2}} x^{-\alpha-1}, & \alpha > -1, \\ 0, & \alpha \leq -1. \end{cases} \quad (1.17)$$

Proof. We will consider two cases.
(i) $\alpha < 0$, using Theorem 15, (1.15), (1.7), and (1.9), we obtain

$$\begin{aligned}
(\widetilde{D}_q {}^c\widetilde{D}_{q,0}^{\alpha} f)(x) &= (\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha} f)(x) \\
&= (\widetilde{D}_q \widetilde{I}_{q,0}^{-\alpha+1}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha+1)} q^{\binom{-\alpha}{2}} \widetilde{D}_q(x^{-\alpha}) \\
&= ({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha+1)} q^{\binom{-\alpha}{2}} q^{1+\alpha} \overline{[-\alpha]}_q x^{-\alpha-1} \\
&= ({}^c\widetilde{D}_{q,0}^{\alpha}\widetilde{D}_q f)(x) + \frac{f(0)}{\widetilde{\Gamma}_q(-\alpha)} q^{\binom{-(\alpha+1)}{2}} x^{-\alpha-1}.
\end{aligned}$$

By Theorem 24, the required equalities are valid both for $\alpha \leq -1$ and $-1 < \alpha < 0$.

(ii) $\alpha > 0$, we assume $\alpha = n + \varepsilon,\ n \in \mathbb{N}_0,\ 0 < \varepsilon < 1$, then $\alpha + 1 \in (n+1, n+2)$, by Theorem 15,(1.15),(1.7) and (1.9), we obtain

$$\begin{aligned}
(\widetilde{D}_q {}^c\widetilde{D}^{\alpha}_{q,0} f)(x) &= (\widetilde{D}_q \widetilde{I}^{1-\varepsilon}_{q,0} \widetilde{D}^{n+1}_q f)(x) \\
&= (\widetilde{D}_q \widetilde{I}^{2-\varepsilon}_{q,0} \widetilde{D}^{n+2}_q f)(x) + \frac{\widetilde{D}^{n+1}_q f(0)}{\widetilde{\Gamma}_q(2-\varepsilon)} q^{\binom{1-\varepsilon}{2}} \widetilde{D}_q (x^{1-\varepsilon}) \\
&= ({}^c\widetilde{D}^{\alpha+1}_{q,0} f)(x) + \frac{\widetilde{D}^{n+1}_q f(0)}{\widetilde{\Gamma}_q(2-\varepsilon)} q^{\binom{1-\varepsilon}{2}} q^{\varepsilon} \overline{[1-\varepsilon]}_q x^{-\varepsilon} \\
&= ({}^c\widetilde{D}^{\alpha+1}_{q,0} f)(x) + \frac{\widetilde{D}^{n+1}_q f(0)}{\widetilde{\Gamma}_q(n+1-\alpha)} q^{\binom{n-\alpha}{2}} x^{n-\alpha}.
\end{aligned}$$

■

Theorem 26 ([7]) *Let* $\alpha \in (N-1, N]$. *Then for some constants* $c_i \in \mathbb{R}$, $i = 0, 1, ..., N$, *the following equality holds*

$$(\widetilde{I}^{\alpha}_{q,0} {}^c\widetilde{D}^{\alpha}_{q,0} f)(x) = f(x) + c_0 + c_1 x + c_2 x^2 + ... + c_{N-1} x^{N-1}.$$

Proof. By (1.15), (1.7) and (1.8), we have

$$\begin{aligned}
(\widetilde{I}^{\alpha}_{q,0} {}^c\widetilde{D}^{\alpha}_{q,0} f)(x) &= (\widetilde{I}^{\alpha}_{q,0} \widetilde{I}^{N-\alpha}_{q,0} \widetilde{D}^{N}_q f)(x) \\
&= (\widetilde{I}^{N}_{q,0} \widetilde{D}^{N}_q f)(x) \\
&= \widetilde{I}^{N-1}_{q,0} ((\widetilde{D}^{N-1}_q f)(x) - (\widetilde{D}^{N-1}_q f)(0)) \\
&= (\widetilde{I}^{N-\alpha}_{q,0} \widetilde{D}^{N-1}_q f)(x) - \frac{q^{\binom{N-1}{2}} (\widetilde{D}^{N-1}_q f)(0)}{\overline{[N-1]}_q!} x^{N-1} \\
&= (\widetilde{I}^{N-2}_{q,0} \widetilde{D}^{N-2}_q f)(x) - \frac{q^{\binom{N-2}{2}} (\widetilde{D}^{N-2}_q f)(0)}{\overline{[N-2]}_q!} x^{N-2} \\
&\quad - \frac{q^{\binom{N-1}{2}} (\widetilde{D}^{N-1}_q f)(0)}{\overline{[N-1]}_q!} x^{N-1} \\
&= ... \\
&= f(x) - \sum_{K=0}^{N-1} \frac{q^{\binom{K}{2}} (\widetilde{D}^{K}_q f)(0)}{\overline{[K]}_q!} x^{K}.
\end{aligned}$$

■

Chapter 2

Nonlinear q-symmetric fractional differential equations of Riemann-Liouville type

In this chapter, we deal with the following nonlinear q-symmetric integral boundary value problem of nonlinear q-symmetric fractional differential equations

$$(\widetilde{D}_{q,0}^{\alpha}u)(t) + f(q^{-\alpha}t, u(q^{-\alpha}t)) = 0, \quad t \in (0, q^{\alpha}), \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \mu(\widetilde{I}_{q,0}^{\beta}u)(\eta), \tag{2.2}$$

where $q \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta \leq 2,\ 0 < \eta < 1$ and $\mu > 0$ is a parameter, $\widetilde{D}_{q,0}^{\alpha}$ is the q-symmetric derivative of Riemann-Liouville type of order α, $f : [0,1] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ is continuous.

Based upon a contraction mapping principle, Schauder's and nonlinear alternative Leray–Schauder's fixed point theorems. We obtain three various results of the existence and uniqueness about the boundary value problem (2.1)-(2.2).

2.1 A few theorems of fixed point

In this section we have different tools of the theory of functional analysis used by following contraction principle of Banach, equicontinuity, theorem of Schauder, theorem of d'Arzela Ascoli... etc.

Definition 27 *Let (E, d) a complete metric space and $F : E \to E$ a continue mapp.*

i) We say that $u \in E$ *is a fixed point of* F *if* $f(u) = u$.
ii) We say that F *is contractante if* F *lipschiz raport* $0 < L < 1$, *i.e there is* $0 < L < 1$, *such that*

$$\forall u, v \in E, d(F(u), F(v)) \leq Ld(u, v),\ 0 < L < 1.$$

Definition 28 (completely continuous) *Let* X *and* Y *two Banach space and* $F : X \to Y$ *a mapp define* X *a value in* F. *We say that* F *is completely continuous if she is continuous and transform any bounded of* Y *in a relatively compact set in* Y. F *is said to be compact if* $F(X)$ *is relatively compact in* Y.

Theorem 29 (Banach see [9]) *Let* X *a Banach space, and a contractant operator* $F : X \to X$. *So* F *admits a unique fixed point.*

$$i.e\ \exists! u \in X \text{ such that } Fu = u.$$

Theorem 30 (Leray Schauder Alternative see [9]) *Let* X *a Banach space,* C *a subset convex in* X, U *is a open subset in* C *and* $0 \in U$. *Assume that* $F : \overline{U} \to C$ *a continous and compact operator* ($F(\overline{U})$ *is relatively compact of* C *). So*

i) F *admits a fixed point of* $\overline{U}$. *or*
ii) There is a $u \in \partial U$ *and* $\lambda \in (0, 1)$ *with* $u = \lambda F(u)$.

Lemma 31 (Krasnoselskii [7]) *Let* E *be a Banach space, and let* $P \subset E$ *be a cone. Assume* Ω_1, Ω_2 *are open subsets of* E *with* $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, *and let* $F : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ *be a completly continous operator such that*

$$\begin{aligned} \|Fu\| &\geq \|u\|, \quad u \in P \cap \partial\Omega_1, \quad \text{and} \\ \|Fu\| &\leq \|u\|, \quad u \in P \cap \partial\Omega_2. \end{aligned}$$

Then F *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 32 (Ascoli-Arzelà [9]) *Let* A *a subset of* $C(J, E)$; A *is relatively compact in* $C(J, E)$ *if and only if the following conditions are verified:*
i) The set A *is bounded. i.e there is a constant* $K > 0$ *such that:*

$$\|f\| \leq K \text{ for evry } x \in J \text{ and } f \in A.$$

ii) The set A *is equicontinuity. i.e for evry* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$|t_1 - t_2| < \delta \Longrightarrow \|f(t_1) - f(t_2)\| \leq \varepsilon \text{ for evry } t_1, t_2 \in J \text{ and } f \in A.$$

iii) for evry $x \in J$ *the set* $\{f(x), f \in A\} \subset E$ *is relatively compact.*

2.2 Fundamental lemmas

We now present some lemmas which will be useful to achieve our main results.

Lemma 33 ([7]) *Let* $y \in C[0,1]$, *and*

$$M = \widetilde{\Gamma}_q(\alpha+\beta) - \mu\eta^{\alpha+\beta-1}q^{\binom{\beta}{2}+(\alpha-1)\beta}\widetilde{\Gamma}_q(\alpha) > 0.$$

Then the unique solution of the boundary value problem

$$(\widetilde{D}^{\alpha}_{q,0}u)(t) + y(q^{-\alpha}t) = 0, \quad t \in (0,q^{\alpha}),\ 1 < \alpha \leq 2, \tag{2.3}$$

with the boundary condition

$$u(0) = 0,\ u(1) = \mu\widetilde{I}^{\alpha}_{q,0}u(\eta),\ 0 < \beta \leq 2,\ 0 < \eta < 1, \tag{2.4}$$

is given by

$$u(t) = \int_0^1 G(t,s)y(q^{-1}s)\widetilde{d}_q s, \quad t \in [0,1], \tag{2.5}$$

where

$$G(t,s) = g(t,s) + \frac{\mu t^{\alpha-1}}{M}H(t,s), \tag{2.6}$$

and

$$g(t,s) = \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\begin{cases} t^{\alpha-1}\overline{(1-s)}^{(\alpha-1)} - \overline{(t-s)}^{(\alpha-1)}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}\overline{(1-s)}^{(\alpha-1)}, & 0 \leq t < s \leq 1, \end{cases} \tag{2.7}$$

$$H(t,s) = \eta^{\alpha+\beta-1}q^{\binom{\alpha+\beta}{2}-\beta}\begin{cases} \overline{(1-s)}^{(\alpha-1)} - \overline{(t-\eta^{-1}q^{-\beta}s)}^{(\alpha+\beta-1)}, & 0 \leq s \leq \eta q^{\beta}, \\ \overline{(1-s)}^{(\alpha-1)}, & \eta q^{\beta} < s \leq 1. \end{cases} \tag{2.8}$$

Proof. In view of theorem 22 , we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\int_0^t \overline{(t-s)}^{(\alpha-1)}y(q^{-1}s)\widetilde{d}_q s, t \in [0,1], \tag{2.9}$$

for some constant $c_1, c_2 \in \mathbb{R}$. since $u(0) = 0$, we have $c_2 = 0$.
Using lemma 17, theorem 14, we have

$$\begin{aligned}(I^{\beta}_{q,0}u)(t) &= c_1 I^{\beta}_{q,0}t^{\alpha-1} - (I^{\alpha+\beta}_{q,0}y)(q^{-\alpha}t) \\ &= c_1\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\beta}{2}+(\alpha-1)\beta}t^{\alpha+\beta-1} \\ &\quad -\frac{1}{\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\int_0^t \overline{(t-s)}^{(\alpha+\beta-1)}y(q^{\beta-1}s)\widetilde{d}_q s.\end{aligned}$$

From the boundary condition $u(1) = \mu \widetilde{I}_{q,0}^{\alpha} u(\eta)$, we get

$$\begin{aligned} c_1 &= \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{M}\Big(\frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^1 \overline{(1-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &\quad - \frac{\mu}{\widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \int_0^{\eta} \overline{(\eta-s)}^{(\alpha+\beta-1)} y(q^{\beta-1}s)\widetilde{d}_q s\Big). \end{aligned}$$

Hence

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}\widetilde{\Gamma}_q(\alpha+\beta)}{M}\Big(\frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^1 \overline{(1-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &\quad - \frac{\mu}{\widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \int_0^{\eta} \overline{(\eta-s)}^{(\alpha+\beta-1)} y(q^{\beta-1}s)\widetilde{d}_q s\Big) \\ &\quad - \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^t \overline{(t-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &= \frac{t^{\alpha-1}}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^1 \overline{(1-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &\quad - \frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}} \int_0^t \overline{(t-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &\quad + \frac{\mu t^{\alpha-1}\eta^{\alpha+\beta-1} q^{\binom{\alpha+\beta}{2}-\beta}}{M} \int_0^1 \overline{(1-s)}^{(\alpha-1)} y(q^{-1}s)\widetilde{d}_q s \\ &\quad - \frac{\mu t^{\alpha-1}}{M} q^{\binom{\alpha+\beta}{2}} \int_0^{\eta} \overline{(\eta-s)}^{(\alpha+\beta-1)} y(q^{\beta-1}s)\widetilde{d}_q s \\ &= \int_0^1 g(t,s) y(q^{-1}s)\widetilde{d}_q s + \frac{\mu t^{\alpha-1}}{M} \int_0^1 H(t,s) y(q^{-1}s)\widetilde{d}_q s \\ &= \int_0^1 G(t,s) y(q^{-1}s)\widetilde{d}_q s. \end{aligned}$$

■

According to the property of being non-increasing of $\overline{(t-s)}^{(\alpha)}$ on s, we may easily obtain lemma34 and Lemma 35 as follows.

Lemma 34 ([7]) *The function $g(t,s)$ and $H(\eta,s)$ satisfy the following properties:*
(i) $g(t,s) \geq 0, g(t,s) \leq g(s,s), 0 \leq t,s \leq 1$.
(ii) $H(\eta,s) \geq 0, 0 \leq s \leq 1$.

Lemma 35 ([7]) *The function $G(t,s)$ satisfies the following properties:*
(i) G is a continous function and $G(t,s) \geq 0, (t,s) \in [0,1] \times [0,1]$.
(ii) There exists a positive function $\rho \in C((0,1),(0,+\infty))$ such that

$$max_{0\leq t\leq 1} G(t,s) \leq g(s,s) + \frac{\mu}{M} H(\eta, s) =: \rho(s), s \in (0,1).$$

Define the operator $F : P \rightarrow X$ as follows:

$$(Fu)(t) = \int_0^1 G(t,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s. \tag{2.10}$$

It follows from the non-negativenes and continuity of G and f that the operator $F : P \rightarrow X$ satisfies $F(P) \subset P$ and is completely continous.

2.3 Results of existence and uniqueness

In this section, we will prove the existence and uniqueness of the solution of problem (2.1)-(2.2) in space $C([0,1],\mathbb{R}^+)$, we use contraction principle of Banach.

Let $X = C[0,1]$ be a Banach space endowed with norm $\|u\|_X = \max_{0\leq t\leq 1} |u(t)|$. Define the cone $P \subset \{u \in X : u(t) \geq 0, 0 \leq t \leq 1\}$.

We define $F : C([0,1],\mathbb{R}^+) \rightarrow C([0,1],\mathbb{R}^+)$ by:

$$u \rightarrow Fu(t) = \int_0^1 G(t,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s. \tag{2.11}$$

and

$$w = \frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^{\beta} \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}. \tag{2.12}$$

Theorem 36 *Let $f : [0,1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a continuous function hold: there exist a constant $L_1 > 0$ such that*

$(\mathbf{H}_1)$ $\left| f\left(q^{-1}s, u(q^{-1}s)\right) - f\left(q^{-1}s, v(q^{-1}s)\right)\right| \leq L_1 |u - v|, \forall s \in [0,1]$ *and* $\forall u, v \in \mathbb{R}$

and

$$L_1 w < 1, \tag{2.13}$$

the problem (2.1)-(2.2) admits unique solution in $[0,1]$.

Proof. To begin the proof, we will transform the problem (2.1)-(2.2) into a fixed point problem $u = Fu$ and Fu define by :

$$Fu(t) = \int_0^1 G(t,s) f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s.$$

Because the problem (2.1)-(2.2) is equivalent to the fractional integral equation (2.10), the fixed points of F are solutions of the problem (2.1)-(2.2).
Let $u, v \in C[0,1]$, for all $t \in [0,1]$ we have:

$$\begin{aligned} |Fu(t) - Fv(t)| &= \left| \int_0^1 G(t,s) \left[f\left(q^{-1}s, u\left(q^{-1}s\right)\right) - f\left(q^{-1}s, v\left(q^{-1}s\right)\right) \right] \widetilde{d}_q s \right| \\ &\leq \int_0^1 G(t,s) \left| f\left(q^{-1}s, u\left(q^{-1}s\right)\right) - f\left(q^{-1}s, v\left(q^{-1}s\right)\right) \right| \widetilde{d}_q s \\ &\leq \int_0^1 G(t,s) \left| u\left(q^{-1}s\right) - v\left(q^{-1}s\right) \right| \widetilde{d}_q s, \end{aligned}$$

Then

$$\begin{aligned} \|Fu - Fv\|_\infty &\leq L_1 \|u - v\|_\infty \int_0^1 G(t,s)\, ds \\ &\leq L_1 \|u - v\|_\infty \left(\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^\beta \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)} \right) \\ &\leq L_1 w \|u - v\|_\infty . \end{aligned}$$

This implies that by (2.13), F is a contraction operator.
Using Banach contraction principle, we deduce that F admits unique fixed point which is the unique solution of the problem (2.1)-(2.2). ■

The next existence result is based on the Krasnoselskii fixed point theorem 31.

Theorem 37 ([7]) *Let $f(t,u)$ be a nonnegative continuous function on $[0,1] \times \mathbb{R}^+$. In addition, we assume that:*
$(\mathbf{H}_2)$ *There exists a positive constant r_1 such that*

$$f(t,u) \geq k r_1, \quad \textit{for} \;\; (t,u) \in [\tau_1, \tau_2] \times [0, r_1],$$

$$\textit{where } \tau_1 = q^{m_3}, \tau_2 = q^{m_4} \;\textit{ with } m_3, m_4 \in \mathbb{N}_0, m_3 > m_4 > 0, \;\textit{ and}$$
$$k \geq \left(q^{-1} \int_{q\tau_1}^{q\tau_2} (g(s,s) + \frac{\mu s^{\alpha-1}}{M} H(\eta, s) \widetilde{d}_q s \right)^{-1} .$$

($\mathbf{H}_3$) *There exists a positive constant r_2 with $r_2 > r_1$ such that*

$$f(t,u) \leq Lr_2, \quad \textit{for } (t,u) \in [0,1] \times [0,r_2],$$

where

$$L = \left(\int_0^1 \frac{q^{\binom{\alpha}{2}} \overline{(1-s)}^{(\alpha-1)}}{\widetilde{\Gamma}_q(\alpha)} + \frac{\mu}{M} H(\eta,s) \widetilde{d}_q s \right)^{-1}.$$

Then the boundary value problem (2.1), (2.2) *has at least one positive solution u_0 satisfying*

$$0 < r_1 \leq \|u_0\|_X \leq r_2.$$

Proof. By Lemma 34, we obtain $max_{0 \leq t \leq 1} g(t,s) = g(s,s)$.
Let $\Omega_1 = \{u \in X : \|u\|_X < r_1\}$. For any $u \in X \cap \partial\Omega_1$, according to (H_2), we have

$$\begin{aligned}
\|Tu\|_X &= \max_{0 \leq t \leq 1} \Big(\int_0^1 g(t,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s \\
&\quad + \int_0^1 \frac{\mu t^{\alpha-1}}{M} H(\eta,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s \Big) \\
&\geq \int_0^1 [g(s,s) + \frac{\mu (s)^{\alpha-1}}{M} H(\eta,s)] f(q^{-1}s, u(q^{-1}s) \widetilde{d}_q s \\
&= (1-q^2) \left[\sum_{k=0}^{\infty} q^{2k} g(q^{2k+1}, q^{2k+1}) + \frac{\mu (q^{2k+1})^{\alpha-1}}{M} H(\eta, q^{2k+1}) \right] f(q^{2k}, u(q^{2k})) \\
&= \int_0^1 \left[g(qs,qs) + \frac{\mu (sq)^{\alpha-1}}{M} H(\eta,qs) \right] f(s,u(s)) \widetilde{d}_{q^2} s \\
&\geq kr_1 \int_0^1 \left[g(qs,qs) + \frac{\mu (sq)^{\alpha-1}}{M} H(\eta,qs) \right] d_{q^2} s \\
&= q^{-1} k r_1 \int_{q\tau_1}^{q\tau_2} [g(s,s) + \frac{\mu (s)^{\alpha-1}}{M} H(\eta,s)] d_{q^2} s \\
&= r_1 \\
&= \|u\|_X
\end{aligned}$$

Let $\Omega_2 = \{u \in X : \|u\|_X < r_2\}$. For any $u \in X \cap \partial\Omega_2$, according to (H_3), we have

$$\begin{aligned}
\|Tu\|_x &= \max_{0 \leq t \leq 1} \textstyle\int_0^1 G(t,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s \\
&\leq Lr_2 \textstyle\int_0^1 \rho(s) \widetilde{d}_q s \\
&\leq \|u\|_X = r_2.
\end{aligned}$$

Now, an application of lemma 31 concludes the proof. ■

The last existence result is based on the Leray-Schauder Alternative fixed point theorem 30.

Theorem 38 *Let $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function such that*
$(\mathbf{H}_4)$ *There is a continuous function non-decroissante $\psi : [0;\infty) \to [0;\infty)$, and a function $P \in C([0,1], \mathbb{R}^+)$ such that*

$$\left|f(q^{-1}s, u(q^{-1}s))\right| \leq P(t)\psi(|u|), \text{ for evry } s \in [0,1], u \in \mathbb{R}.$$

$(\mathbf{H}_5)$ *There is a constant $L_2 > 0$ such that*

$$\frac{L_2}{\psi(L_2)\left\|P\right\|\left(\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^{\beta}\frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}\right)} > 1.$$

So the problem (2.1)-(2.2) has at least one solution on $C[0,1]$.

Proof. Let $\Omega = \{u \in X : \|u\|_X < L_2\}$ and L_2 is taked by (H_5).
It is esay that the subset Ω is closed and convex, we new proved that F satisfed the conditions of Leray-Schauder Alternative fixed point theorem 30 with

$$F : C([0,1], \mathbb{R}^+) \to C([0,1], \mathbb{R}^+)$$

$$u \to Fu\,(t) = \int_0^1 G\,(t,s)\, f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s.$$

This could be proved through three steps :
Step 1.F is a continuous operator. Let $(u_n)_{n\in\mathbb{N}}$ be real sequence such that $\lim\limits_{n\to\infty} u_n = u$ in $C[0,1]$. Then for each $t \in C[0,1]$.

$$\begin{aligned}
|Fu_n\,(t) - Fu\,(t)| &= \int_0^1 G\,(t,s)\left|f\left(q^{-1}s, u_n\left(q^{-1}s\right)\right) - f\left(q^{1}s, u\left(q^{-1}s\right)\right)\right| \widetilde{d}_q s \\
&\leq \left\|f\left(q^{-1}s, u_n\left(q^{-1}s\right)\right) - f\left(q^{-1}s, u\left(q^{-1}s\right)\right)\right\|_\infty \int_0^1 G\,(t,s)\, \widetilde{d}_q s \\
&\leq \left(\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \frac{\eta q^{\beta}\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}\right)\left\|f\left(q^{-1}s, u_n\left(q^{-1}s\right)\right) - f\left(q^{-1}s, u\left(q^{-1}s\right)\right)\right\|_\infty
\end{aligned}$$

For each $t \in C[0,1]$, the function $s \to G(t;s)$ is integrable on $[0,1]$, then the lebesgue dominated convergence theorem imply that :

$$|Fu_n\,(t) - Fu\,(t)| \to 0, \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|Fu_n - Fu\|_\infty = 0.$$

Consequently, F is continuous.

Step 2. F transformed Ω by a bounded set of $C([0,1],\mathbb{R}^+)$,

$$\begin{aligned}
|Fu\,(t)| &\leq \max_{0\leq t\leq 1}\{\int_0^1 G\,(t,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\}\widetilde{d}_q s \\
&\leq \max_{0\leq t\leq 1} P(t)\psi(|u|)\int_0^1 G\,(t,s)\,\widetilde{d}_q s \\
&\leq \max_{0\leq t\leq 1} P(t)\psi(|u|)\left(\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^\beta \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}\right),
\end{aligned}$$

consequently,

$$\|Fu\| \leq \psi(u)\,\|P\|\left(\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^\beta \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}\right).$$

Step 3. F transformed Ω in a set equicontinue of $C([0,1],\mathbb{R}^+)$.

Let $f \in C([0,1],\mathbb{R}^+)$, τ_1, $\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $u \in \Omega$. So we have

$$\begin{aligned}
|F(u)\,(\tau_2) - F(u)\,(\tau_1)| &\leq \int_0^1 G\,(\tau_2,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right| \\
&\quad - \int_0^1 G\,(\tau_1,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\widetilde{d}_q s \\
&\leq \int_0^1 [G\,(\tau_2,s) - G\,(\tau_1,s)]\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\widetilde{d}_q s \\
&\leq P(t)\psi(|u|)\int_0^1 [G\,(\tau_2,s) - G\,(\tau_1,s)]\widetilde{d}_q s.
\end{aligned}$$

For $0 \leq s \leq \tau_1 \leq \tau_2 \leq 1$, there exist a constant $K > 0$ we have :

$$\int_0^1 [G\,(\tau_2,s) - G\,(\tau_1,s)]\widetilde{d}_q s \leq (\tau_2^{\alpha-1} - \tau_1^{\alpha-1})\left[\frac{1}{\widetilde{\Gamma}_q(\alpha+1)} + \frac{K}{M}\right].$$

In the same way, for $0 \leq \tau_1 \leq s \leq \tau_2 \leq 1$ or $0 \leq \tau_1 \leq \tau_2 \leq s \leq 1$, we have:

$$\int_0^1 [G\,(\tau_2,s) - G\,(\tau_1,s)]\widetilde{d}_q s \leq (\tau_2^{\alpha-1} - \tau_1^{\alpha-1})\left[\frac{1}{\widetilde{\Gamma}_q(\alpha+1)} + \frac{K}{M}\right].$$

Then

$$\begin{aligned}
|F(u)(\tau_2) - F(u)(\tau_1)| &\leq \int_0^1 [G(\tau_2, s) - G(\tau_1, s)] \left| f\left(q^1 s, u\left(q^{-1}s\right)\right)\right| \widetilde{d}_q s \\
&\leq P(t)\psi(|u|) \int_0^1 [G(\tau_2, s) - G(\tau_1, s)] \widetilde{d}_q s. \\
&\leq P(t)\psi(|u|)(\tau_2^{\alpha-1} - \tau_1^{\alpha-1}) \left[\frac{1}{\widetilde{\Gamma}_q(\alpha+1)} + \frac{K}{M}\right].
\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zeros. Consequently, by alternative non-linear Leray-Schauder theorem 30. Then F has a fixed point which is a solution of problem (2.1)-(2.2). ■

2.4 An example

Example 39 *Consider the following nonlinear fractional q-symmetric differential equation*

$$\begin{cases} (\widetilde{D}^{\alpha}_{q,0}u)(t) + \left[\sin(\sqrt{2}t) + \frac{u^2}{u+1}\right] = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \mu(\widetilde{I}^{\beta}_{q,0}u)(\eta). \end{cases} \tag{2.14}$$

With $\alpha = \frac{1}{2}$, $\beta = 1$, $\eta = \frac{1}{2}$, $q = \frac{1}{2}$ *and*

$$f(q^{-\alpha}t, u(q^{-\alpha}t)) = \sin(\sqrt{2}t) + \frac{u^2(q^{-\alpha}t)}{u(q^{-\alpha}t)+1}.$$

As $sin(\sqrt{2}t)$ *are continuous positive functions* $\forall t \in (0,1)$ *the function* f *is jointly continuous. For any* $u, v \in \mathbb{R}$. *Then*

$$\begin{aligned} \left|f(q^{-\alpha}t, u(q^{-\alpha}t)) - f(q^{-\alpha}t, v(q^{-\alpha}t))\right| &= \left|\sin(\sqrt{2}t) + \frac{u^2}{u+1} - \sin(\sqrt{2}t) + \frac{v^2}{v+1}\right| \\ &= \left|\sin(\sqrt{2}t)\right| \left|\frac{u^2}{u+1} - \frac{v^2}{v+1}\right| \\ &\leq |u-v|. \end{aligned}$$

So we have

$$\begin{aligned} w &= \frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^{\beta} \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)} \\ &= 0.535714. \end{aligned}$$

Hence, the condition (2.13) *is satisfied with* $L_1 = 1$.

$$\begin{aligned} L_1 w &= L_1 \left[\frac{\widetilde{\Gamma}_q(\alpha)}{\widetilde{\Gamma}_q(\alpha+1)} - \eta q^{\beta} \frac{\widetilde{\Gamma}_q(\alpha+\beta)}{\widetilde{\Gamma}_q(\alpha+\beta+1)}\right] \\ &= 1 \times 0.535714 \\ &< 1. \end{aligned}$$

So by theorem 36 the problem (2.14) has a unique solution in $[0,1]$.

Chapter 3

Nonlinear q-symmetric fractional differential equations caputo type

In this chapter, we deal with the following nonlocal q-symmetric integral boundary value problem of nonlinear fractional q-symmetric differential equations

$$({}^{c}\widetilde{D}_{q,0}^{\alpha}u)(t)+f(q^{-\alpha}t,u(q^{-\alpha}t))=0,\quad t\in(0,q^{\alpha}), \tag{3.1}$$

$$u(0)=0,\quad u(1)=\mu(\widetilde{I}_{q,0}^{\beta}u)(\eta), \tag{3.2}$$

where $q\in(0,1)$, $1<\alpha\leq 2$, $0<\beta\leq 2$, $0<\eta<1$ and $\mu>0$ is a parameter, ${}^{c}\widetilde{D}_{q,0}^{\alpha}$ is the q-symmetric fractional derivative of Caputo type of order α, $f:[0,1]\times\mathbb{R}^{+}\longrightarrow\mathbb{R}^{+}$ is continuous.

Based upon a contraction mapping principle, Schauder's and nonlinear alternative Leray–Schauder's fixed point theorems. We obtain two various results of the existence and uniqueness about the boundary value problem (3.1)-(3.2).

3.1 Fundamental lemmas

First, we will define the integral solution of the problem (3.1)-(3.2).

Lemma 40 *Let* $y \in C[0,1]$, *and*

$$M = 1 + \frac{\mu}{\widetilde{\Gamma}_q(\beta)} q^{\binom{\beta}{2}} \eta^{\beta} > 0,$$

Then the unique solution of the boundary value problem

$$({}^{c}\widetilde{D}_{q,0}^{\alpha} u)(t) + y(q^{-\alpha} t) = 0, \quad t \in (0, q^{\alpha}), \ 1 < \alpha \leq 2, \tag{3.3}$$

with the boundary condition

$$u(0) = 0, \ u(1) = \mu \widetilde{I}_{q,0}^{\alpha} u(\eta), \ 0 < \beta \leq 2, \ 0 < \eta < 1, \tag{3.4}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(q^{-1}s) \widetilde{d}_q s, \quad t \in [0,1], \tag{3.5}$$

such that G *is a continous function and* $G(t,s) \geq 0, \ (t,s) \in [0,1] \times [0,1]$ *where*

$$\int_0^1 G(t,s) \widetilde{d}_q s \preceq \frac{\mu}{M \widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta}, \tag{3.6}$$

Define the operator $F : P \to X$ as follows:

$$(Fu)(t) = \int_0^1 G(t,s) f(q^{-1}s, u(q^{-1}s)) \widetilde{d}_q s. \tag{3.7}$$

It follows from the non-negativenes and continuity of G and f that the operator $F : P \to X$ satisfies $F(P) \subset P$ and is completely continous.

3.2 Results of existence and uniqueness

Now, we will prove our first existence result for the problem (3.1)-(3.2) which is based on Banach fixed point theorem.

We impose the following hypotheses :
($\mathbf{H}_1$) $f : [0,1] \to \mathbb{R}^+$ is a continuous function.
($\mathbf{H}_2$) For all $1 < \alpha \leq 2$, there exist a constant $\lambda > 0$ such that :

$$\left| f\left(q^{-1}s, u(q^{-1}s)\right) - f\left(q^{-1}s, v(q^{-1}s)\right) \right| \leq \lambda \left| u - v \right|, \ \forall s \in [0,1] \text{ and } \forall u, v \in \mathbb{R} \tag{3.8}$$

($\mathbf{H}_3$) There is a continuous function non-decroissante $\psi : [0;\infty) \to [0;\infty)$, and a function $P \in C([0,1], \mathbb{R}^+)$ such that

$$\left| f(q^{-1}s, u(q^{-1}s)) \right| \leq P(t)\psi(|u|), \text{ for evry } s \in [0,1], \ u \in \mathbb{R}. \tag{3.9}$$

Theorem 41 *Assume the hypotheses* ($\mathbf{H}_1$), ($\mathbf{H}_2$) *hold. We give* $1 < \alpha \leq 2, \ 0 < \beta \leq 2, \ 0 < \eta < 1$ *such that*

$$\lambda \left(\frac{\mu}{M \widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta} \right) < 1. \tag{3.10}$$

Then the problem (3.1)-(3.2) admits a unique solution on $[0,1]$.

Proof. To begin the proof, we will transform the problem (3.1)-(3.2) into a fixed point problem $u = Fu$ and Fu define by :

$$Fu\left(t\right) = \int_0^1 G\left(t,s\right) f\left(q^{-1}s, u\left(q^{-1}s\right)\right) \widetilde{d}_q s.$$

Because the problem (3.1)-(3.2) is equivalent to the fractional integral equation (3.7), the fixed points of F are solutions of the problem (3.1)-(3.2).
Let $u, v \in C\left[0,1\right]$, There for all $t \in [0,1]$:

$$\begin{aligned} \left| Fu\left(t\right) - Fv\left(t\right) \right| &= \left| \int_0^1 G\left(t,s\right) \left[f\left(q^{-1}s, u\left(q^{-1}s\right)\right) - f\left(q^{-1}s, v\left(q^{-1}s\right)\right) \right] \widetilde{d}_q s \right| \\ &\leq \int_0^1 G\left(t,s\right) \left| f\left(q^{-1}s, u\left(q^{-1}s\right)\right) - f\left(q^{-1}s, v\left(q^{-1}s\right)\right) \right| \widetilde{d}_q s \\ &\leq \int_0^1 G\left(t,s\right) \left| u\left(q^{-1}s\right) - v\left(q^{-1}s\right) \right| \widetilde{d}_q s, \end{aligned}$$

Then

$$\begin{aligned}\|Fu-Fv\|_\infty &\leq \lambda\|u-v\|_\infty\int_0^1 G(t,s)\,ds\\ &\leq \lambda\|u-v\|_\infty\left[\frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}\right]\\ &\leq \lambda\|u-v\|_\infty .\end{aligned}$$

This implies that by (3.10), F is a contraction operator.
Using Banach contraction principle, we deduce that F admits unique fixed point which is the unique solution of the problem (3.1)-(3.2). ∎

The next existence result is based on the nonlinear alternative of Leray-Schauder type.

Theorem 42 *Assume the hypotheses* $(\mathbf{H}_1)$, $(\mathbf{H}_3)$ *hold. We give* $1<\alpha\leq 2,\ 0<\beta\leq 2,\ 0<\eta<1$ *and* $\rho>0$ *such that*

$$\frac{\rho}{\psi(\rho)\|P\|\left(\frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}\right)}>1$$

Then the problem (3.1)-(3.2) has at least one solution on $C[0,1]$.

Proof. Let $\Omega_1=\{u\in X:\|u\|_X<\rho\}$.
It is esay that the subset Ω_1 is closed and convex, we new proved that F satisfed the conditions of the Leray-Schauder Alternative fixed point theorem 30 with

$$F:C([0,1],\mathbb{R}^+)\to C([0,1],\mathbb{R}^+)$$

$$u\to Fu(t)=\int_0^1 G(t,s)\,f\left(q^{-1}s,u\left(q^{-1}s\right)\right)\widetilde{d}_qs.$$

This could be proved through three steps :
Step 1.F is a continuous operator. Let $(u_n)_{n\in\mathbb{N}}$ be real sequence such that $\lim\limits_{n\to\infty}u_n=u$ in $C[0,1]$. Then for each $t\in C[0,1]$.

$$\begin{aligned}|Fu_n(t)-Fu(t)| &= \int_0^1 G(t,s)\left|f\left(q^{-1}s,u_n\left(q^{-1}s\right)\right)-f\left(q^{1}s,u\left(q^{-1}s\right)\right)\right|\widetilde{d}_qs\\ &\leq \left\|f\left(q^{-1}s,u_n\left(q^{-1}s\right)\right)-f\left(q^{-1}s,u\left(q^{-1}s\right)\right)\right\|_\infty\int_0^1 G(t,s)\,\widetilde{d}_qs\\ &\leq \left(\frac{\mu q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}}{M\widetilde{\Gamma}_q(\alpha+\beta)}\right)\left\|f\left(q^{-1}s,u_n\left(q^{-1}s\right)\right)-f\left(q^{-1}s,u\left(q^{-1}s\right)\right)\right\|_\infty .\end{aligned}$$

For each $t \in C[0,1]$, the function $s \to G(t,s)$ is integrable on $[0,1]$, then the lebesgue dominated convergence theorem imply that :

$$|Fu_n(t) - Fu(t)| \to 0, \text{ as } n \to \infty,$$

and hence

$$\lim_{n\to\infty} \|Fu_n - Fu\|_\infty = 0.$$

Consequently, F is continuous.

Step 2. F transformed Ω_1 by a bounded set of $C([0,1],\mathbb{R}^+)$,

$$\begin{aligned} |Fu(t)| &\leq \max_{0\leq t\leq 1}\{\int_0^1 G(t,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\}\widetilde{d}_q s \\ &\leq \max_{0\leq t\leq 1} P(t)\psi(|u|)\int_0^1 G(t,s)\,\widetilde{d}_q s \\ &\leq \max_{0\leq t\leq 1} P(t)\psi(|u|)\left(\frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}\right). \end{aligned}$$

Consequently,

$$\|Fu\| \leq \psi(u)\,\|P\|\left(\frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}\right).$$

So $F(\Omega_1)$ is bounded and $F(\Omega_1) \subset \Omega_1$.

Step 3. F transformed Ω_1 in a set equicontinue of $C([0,1],\mathbb{R}^+)$.

Let $f \in C([0,1],\mathbb{R}^+)$, τ_1, $\tau_2 \in [0,1]$ with $\tau_1 < \tau_2$ and $u \in \Omega_1$. So we have

$$\begin{aligned} |F(u)(\tau_2) - F(u)(\tau_1)| &\leq \int_0^1 G(\tau_2,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right| \\ &\quad - \int_0^1 G(\tau_1,s)\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\widetilde{d}_q s \\ &\leq \int_0^1 [G(\tau_2,s) - G(\tau_1,s)]\left|f\left(q^1 s, u\left(q^{-1}s\right)\right)\right|\widetilde{d}_q s \\ &\leq P(t)\psi(|u|)\int_0^1 [G(\tau_2,s) - G(\tau_1,s)]\widetilde{d}_q s. \end{aligned}$$

For $0 \leq s \leq \tau_1 \leq \tau_2 \leq 1$, there exist a constant $K > 0$ we have :

$$\begin{aligned} \int_0^1 [G(\tau_2,s) - G(\tau_1,s)]\widetilde{d}_q s &\leq (\tau_2 - \tau_1)\left[\frac{-1}{\widetilde{\Gamma}_q(\alpha)M}q^{\binom{\alpha}{2}} + \frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)}q^{\binom{\alpha+\beta}{2}}\eta^{\alpha+\beta}\right] \\ &\quad -(\tau_2^\alpha - \tau_1^\alpha)\left[\frac{1}{\widetilde{\Gamma}_q(\alpha)}q^{\binom{\alpha}{2}}\right]. \end{aligned}$$

In the same way, for $0 \leq \tau_1 \leq s \leq \tau_2 \leq 1$ or $0 \leq \tau_1 \leq \tau_2 \leq s \leq 1$, we have:

$$\begin{aligned} \int_0^1 [G(\tau_2, s) - G(\tau_1, s)]\widetilde{d}_q s \;\; &\leq \;\; (\tau_2 - \tau_1)\left[\frac{-1}{\widetilde{\Gamma}_q(\alpha) M} q^{\binom{\alpha}{2}} + \frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta}\right] \\ &\quad -(\tau_2^\alpha - \tau_1^\alpha)\left[\frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}}\right]. \end{aligned}$$

Then

$$\begin{aligned} |F(u)(\tau_2) - F(u)(\tau_1)| \;\; &\leq \;\; \int_0^1 [G(\tau_2, s) - G(\tau_1, s)] \left| f\left(q^1 s, u\left(q^{-1}s\right)\right)\right| \widetilde{d}_q s \\ &\leq \;\; P(t)\psi(|u|)\int_0^1 [G(\tau_2, s) - G(\tau_1, s)]\widetilde{d}_q s. \\ &\leq \;\; P(t)\psi(|u|)(\tau_2 - \tau_1)\left[\frac{-1}{\widetilde{\Gamma}_q(\alpha) M} q^{\binom{\alpha}{2}} + \frac{\mu q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta}}{M\widetilde{\Gamma}_q(\alpha+\beta)}\right] \\ &\quad -P(t)\psi(|u|)(\tau_2^\alpha - \tau_1^\alpha)\left[\frac{1}{\widetilde{\Gamma}_q(\alpha)} q^{\binom{\alpha}{2}}\right]. \end{aligned}$$

As $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zeroa,by Leray-Schauder theorem 30. Then F has a fixed point which is a solution of problem (3.1)-(3.2). ■

3.3 An example

Example 43 *Consider the following nonlinear fractional q-symmetric differential equation*

$$\begin{cases} ({}^{c}\widetilde{D}_{q,0}^{\alpha}u)(t) + \left[\frac{e^{q^{-\alpha}t}}{3}\sin^2(q^{-\alpha}t)u(q^{-\alpha}t)\right] = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \mu(\widetilde{I}_{q,0}^{\beta}u)(\eta). \end{cases} \tag{3.11}$$

With $\alpha = 2$, $\beta = 1$, $\eta = \frac{1}{2}$, $q = \frac{1}{2}$, $t = \frac{1}{2}$, $\mu = 1$ *and*

$$f(q^{-\alpha}t, u(q^{-\alpha}t)) = \frac{e^{q^{-\alpha}t}}{3}\sin^2(q^{-\alpha}t)u(q^{-\alpha}t).$$

As $\sin^2(q^{-\alpha}t)$ *are continuous positive function* $\forall t \in (0,1)$, *the function* f *is jointly continuous. For any* $u, v \in \mathbb{R}$. *Then*

$$\begin{aligned} \left|f(q^{-\alpha}t, u(q^{-\alpha}t)) - f(q^{-\alpha}t, v(q^{-\alpha}t))\right| &= \left|\frac{e^{q^{-\alpha}t}}{3}\sin^2(q^{-\alpha}t)u - \frac{e^{q^{-\alpha}t}}{3}\sin^2(q^{-\alpha}t)v\right| \\ &\leq \frac{1}{3}\left|e^{q^{-\alpha}t}\sin^2(q^{-\alpha}t)\right| |u - v| \\ &\leq \frac{1}{3}|u - v|. \end{aligned}$$

So we have

$$\begin{aligned} w &= \frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta} \\ &= 0.0315. \end{aligned}$$

Hence, the condition (3.10) *is satisfied with* $\lambda = \frac{1}{3}$.

$$\begin{aligned} \lambda w &= \lambda[\frac{\mu}{M\widetilde{\Gamma}_q(\alpha+\beta)} q^{\binom{\alpha+\beta}{2}} \eta^{\alpha+\beta}] \\ &= 0.0105 < 1. \end{aligned}$$

So by theorem 41 the problem (3.1) admits unique solution in $[0,1]$.

Conclusion

In this memory we have presented the basic notations, definitions and properties concerning q-symmetric fractional calculus and some results of existence and uniqueness of solutions to the problem at the limits for q-symmetric fractional differential equations involving Riemann-Liouville and Caputo q-symmetric fractional derivative with boundary and integral conditions.

For our discussion, we have used Schauder, Krasnoselskii, Leray Schauder Alternative fixed point theorems and the Banach contraction principle.

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